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WAVES PRODUCED BY THE ELASTIC IMPACT OF SPHERES ON THICK PLATES

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Abstract—An experimental and theoretical investigation of the elastic normal impact of spheres on thick plates has been carried out. The coefficient of restitution has been measured very carefully by observing the flight times of successive bounces of balls on glass plates and the displacements at the bottom face have been accurately measured with a laser heterodyne optical interferometer. The analysis along the axis of impact has shown two unexpected phenomena, namely the propagation of a longitudinal displacement at the shear wave velocity and an enhancement of the amplitude of the reflected "P" wave at the free surface of the plate. Both of these have been confirmed experimentally.

INTRODUCTION

Two extreme cases of the elastic impact of spheres on plates have so far been considered in the literature. These are first, impacts on plates so thick that they can be considered to be semi-infinite solids. This problem was first considered by Hertz[1] and more recently and in greater detail by Hunter[2]. The second case which has been considered, is that of impacts on plates the thickness of which is very small compared with the distance that a "P" wave can travel during the duration of the impact. This was first studied by Raman[3] who showed that as the ratio γ ($\gamma = 2r_s/h$ where r_s is the radius of the sphere and h is the thickness of the plate) increases, the value of e, the coefficient of restitution of the ball, falls from its initial value, which is slightly less than unity when $\gamma < 0.1$, to almost zero when $\gamma > 3$. For such large values of γ the sphere almost sticks to the plate in what appears to be an inelastic impact. Raman correctly interpreted this "inelasticity" to be the result of flexural elastic waves which are set up in the plate, and his semi-empirical treatment gave good agreement with experimental results over a limited range of γ.

Zener and Feshbach[4], and Zener[5] published a much more detailed analysis of the problem. Their treatment was based on the assumption that the kinetic energy of the sphere was divided between setting up a Hertzian stress field near the region of contact of the two colliding bodies, and the propagation of cylindrically diverging flexural elastic waves in the plate. These flexural waves emanate from the region of contact between the sphere and the plate.

Now, at the beginning of the impact process (i.e. at times less than the time it takes a "P" wave to twice traverse the plate thickness), the stress conditions in the area of contact must be identical to those for a semi-infinite solid, since the sphere does not "know" how thick the plate is. It is only after many reflections of the "P" and "S" waves produced by the impact have taken place at the two free surfaces of the plate, that the concept of "flexural waves" becomes significant. When the time of contact is less than twice the transit time of a "P" wave across the plate thickness, e has the same value as it has for the impact with a semi-infinite solid, since the force between the sphere and the plate will be undisturbed by reflected waves, and it would seem likely that if the thickness is such that only two or three reflections of the elastic waves take place at the surfaces of the plate, the Zener treatment, which is based on fully formed flexural waves, would not give close agreement with observations. However, the theoretical predictions can be made to agree quite well with the experimental observations even for thick plates. This close agreement is, however, largely illusory, since the value of e for such thick plates is close to unity. If one compares what is

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a more relevant parameter $(1-e^2)$, which is the fraction of the initial kinetic energy of the sphere lost in the impact, the differences are, as will be shown later, very much more significant.

In the limiting case $\gamma \to \infty$, the Zener treatment becomes identical to that of Hertz, and $e = 1$. Hunter[2] has shown that this is not quite true since a small energy loss, which results from the emanating elastic waves, always exists. This loss will be referred to as the *Hunter loss.*

In this paper the problem, which is considered in detail, is the impact of spheres on plates which are neither "infinitely thick" (i.e. not thicker than the critical value when a "P" wave has time to traverse twice the plate thickness) nor so thin that the concept of flexural elastic waves is appropriate. The energy losses expected for impacts wih such plates will be considered, but the most interesting aspect of this study has turned out to be the light it has thrown on the propagation and reflection of the spherically divergent waves which were set up by the elastic impact of a sphere.

ELASTIC IMPACT ON A HALF SPACE

The elastic impact on a sphere on an elastic half space was first considered by Hertz[1]. His theory was essentially quasi-static, in that it assumed that the stress fields set up in the sphere and the half space were identical to those produced by static contact between them. The reasons that such a treatment is valid are that the time of contact is long enough for many reflections of elastic waves across the impacting sphere, and that for the half space, the stress field is set up by elastic waves which diverge spherically from the region of contact, such waves result in quasi-static stress fields as they propagate outwards.

Hertz gives expressions for the time of contact between the two bodies and predicts that the coefficient of restitution e will be unity for such impacts. If s is the distance of relative approach of the sphere and the half space, Hertz shows that the force F between the two bodies is given by

$$F = Ks^{3/2}$$

where

$$K = \tfrac{4}{3}r_s^{1/2}E'_pE'_s/(E'_p+E'_s)$$

and

$$E'_p = E_p/(1-\nu_p^2)$$

while

$$E'_s = E_s/(1-\nu_s^2).$$

E_p and ν_p are Young's modulus and Poisson's ratio, respectively, of the half space (or in this case, the plate) and E_s and ν_s are the corresponding quantities for the sphere.

In order to elucidate the physical basis of the wave propagation process which is operative in the impact problem, Hunter[2] considered the simpler problem of the uniform expansion of a spherical cavity of initial radius a in an infinite elastic solid when the cavity is subjected to a step function in pressure, i.e. $p(t) = p_0H(t)$. The dynamic effects die out exponentially with time and the characteristic time τ_c is given by $\tau_c = 2a/c_0$, where $c_0 = (E/\rho)^{1/2}$ is the velocity of rod waves in the elastic solid. The elastic energy stored in the spherical shell between radii r and $r+dr$ decreases rapidly with increasing r, and the diverging "P" wave carries half of the energy of the pulse to infinity, the other half is deposited as stored elastic energy in the medium. As the time of loading increases from step loading to times comparable with τ_c, less and less of the energy is carried away to infinity in the form of elastic waves, and in the limit of "static" loading, no energy is lost in this way. Now, if the pressure in the cavity is suddenly brought back to zero (this is equivalent to applying a second pulse $p(t) = -p_0H(t)$ to it), the elastic energy stored in the medium is carried away to infinity. If instead, the pressure is reduced slowly enough, all the stored elastic energy will be recovered as work done by the walls of the cavity.

In the impact of a sphere on a half space the characteristic time will be of the order of $2R/c_0$, where R is a length comparable to the radius of the sphere. It is thus at least one order of magnitude smaller than the contact time of the impact, and the Hertzian approximation of ignoring the loss of energy by diverging elastic waves is therefore a reasonably sound one.

Now although $\tau_c \ll T_1$, T_1 being the contact time, some energy will be carried away in the form of elastic waves, and Hunter[2] obtained an estimate of the magnitude of this by extending the classical treatment of Lamb[6] to the problem. Lamb considered the wave system produced by a point force acting normally on an elastic half space. Miller and Pursey[7] extended this treatment to cover the response of such a half space to a harmonically oscillating pressure applied normally and uniformly to a small area of the free surface of the half space, and in a later paper[8] they calculated the distribution of energy between the different elastic waves which diverge from the loading area. They found that for a Poisson's ratio $\nu = 0.25$, 67% of the energy was propagated as a Rayleigh surface wave, 26% was in an "S" wave, and only 7% was in the diverging "P" wave. The energy transfer is a result of the phase lag of the oscillating displacement behind the oscillating applied force.

Hunter approximated to the time history of the impact force by a half sine wave, since this has a convenient Fourier transform. Using Miller and Pursey's results he was thus able to calculate the energy loss in the form of elastic waves for an elastic sphere hitting an elastic half space. When a steel sphere hits a glass half space, he derived the relation

$$1 - e^2 = 7.56 \times 10^{-3} V_0^{3/5}$$

where V_0 is measured in cm s^{-1}.

As will be shown later, this expression underestimates the observed losses by a factor of about two, possible reasons for this difference will also be discussed. In order to simplify the analysis, Hunter assumed that the pressure applied was uniform over a mean value of the actual contact area. It turns out that the use of this approximation is not altogether valid because of the hyperbolic nature of the wave equations, and Tsai and Kolsky[9], and Tsai[10] have shown that a tiny peak in the tail of the generated Rayleigh wave, which persists after considerable distances of travel, is directly attributable to the variation in the contact area during the impact. However, this effect is unimportant in finding the fraction of the energy which is propagated.

EXPERIMENTAL

The most direct method of measuring the fractional energy loss on rebound of a steel ball hitting a plate, is to find the height to which the ball rises after it has been dropped on to the plate from a known height h_1. Thus, if the rebound height is h_2 and, if air losses are neglected, the fractional loss in energy is given by $(h_1 - h_2)/h_1 = \Delta W/W$, where W is the kinetic energy of the ball before it hits the plate and ΔW is the energy lost in the impact.

Such measurements have been made in various earlier studies, e.g. Tillett[11]. Tillett carried out the experiments in a darkened room, and illuminated the ball with the light from a bright point source. She obtained a photographic record of the path of the ball using a camera pointed at the ball with its shutter open during the rebounds.

This method is quite effective if what is needed is the value of e, the coefficient of restitution, thus we found that with care, using this method, values of e could be obtained with a relative accuracy of $\pm\frac{1}{4}\%$. If we are interested, however, in the value of $\Delta W/W = 1 - e^2$, and e^2 is greater than 0.95, the relative accuracy falls to about $\pm 10\%$, which is not close enough for meaningful comparisons to be made with the theory.

Table 1. Coefficients of restitution for steel spheres hitting thick glass plates (velocity of impact: 1.1 m s^{-1})

	$2r_s$ (mm)	e (%) $\pm$0.08%	e^2 (%) $\pm$0.16%
57 mm Plate	3	99.33	98.66
	4	99.32	98.65
	6	99.37	98.75
	8	99.28	98.56
	10	99.10	98.20
19 mm Plate	3	99.01	98.02
	4	98.67	97.36
	6	97.72	95.50
	8	96.42	92.97

We found that a more accurate method of measuring **e** was to determine electrically the flight time of successive bounces of a ball on the plate. This was done by fixing a piezoelectric detector on the surface of the plate and using the signals from it to start and stop two microsecond timers. In the absence of air resistance, the ratio of two successive flight times is equal to the ratio of the velocities of approach and rebound of the ball, i.e. **e**, the coefficient of restitution. In carrying out these experiments, allowance was made for air resistance, and since it is easier to achieve high relative accuracy in measurements of time intervals than in measurements of distance, and it is also simpler to carry out a large number of observations, it was found that a much higher relative accuracy of the value of $1-e^2$ could be achieved.

The experiments were carried out by dropping hardened steel balls on to large glass plates. Plates of two different thicknesses, 19 and 57 mm were employed. Glass was chosen as the material for the plates because of its almost perfect elastic response, and because it is readily obtainable with very smooth flat surfaces. (In contrast, Lifshitz and Kolsky[12] found that with metal plates it was extremely difficult to produce really flat surfaces, and that, unless the plate specimens were prepared with enormous care, localized plastic deformations took place at "high points" on the surface, this occurred even when the velocities of impact were very low.)

In the present experiments, the balls were dropped from a height of 60 mm above the plates, the energy losses due to air resistance were here less than 0.5%, so that these losses did not need to be evaluated very accurately. Using the electrical method of measuring **e**, it was found that by taking the mean of 12 measurements, the value of **e** obtained had a probable error of about $\pm$0.08%, this led to a probable error in the value of e^2 of about $\pm$0.16%. Assuming $e^2 = 0.95$, this gives a relative accuracy of $1-e^2$ of about $\pm$3%, which was considered acceptable for comparison with the theoretical predictions.

The plate which was 57 mm thick could be regarded as "infinitely thick" for impacts with the 3 and 4 mm spheres (and in practice, this was so even for spheres of 6 mm in diameter). The values of **e** and e^2 are listed in Table 1, and it may be seen that the values are close to unity, but they are still less than those predicted by Hunter[2] for a perfectly elastic impact. (It should perhaps be mentioned here that Tillett[11] does not record values of **e** greater than 0.985, this may be due to the fact that she did not allow for air resistance.)

The other parameter which we wanted to measure was the motion of the bottom face of a plate which resulted from the impact of a sphere on its top face. For thick plates the magnitude of this displacement is very small indeed, and detectors of extremely high sensitivity must be used. The method that was first tried was to employ a capacitative detector. Although this method can be made to be quite sensitive, it was found, after a few preliminary experiments, that the sensitivity was still insufficient for reliable "point" measurements to be made with impacts on thicker plates, and an even more sensitive optical method of detection based on a laser heterodyne interferometer system was employed instead.

The general arrangement of the interferometer is illustrated in Fig. 1.† The physical

† This interferometer was designed and built by Dr J. Goodbread at ETH Zurich.

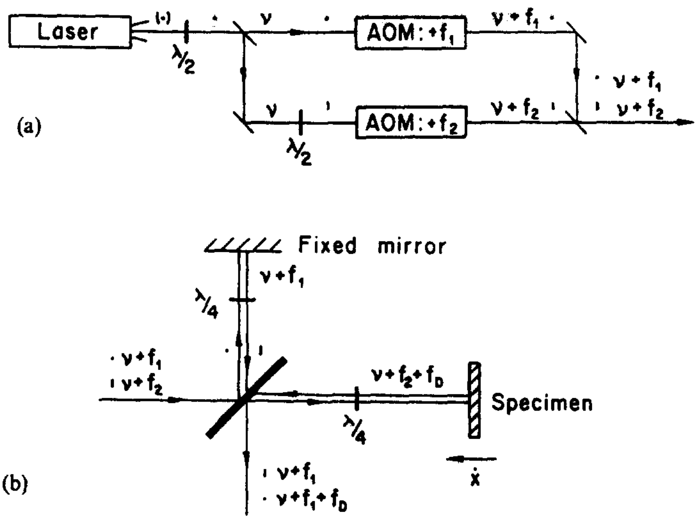

Fig. 1. Optical arrangements for Doppler interferometer: (a) beam formation (·, vertical; |, horizontal polarization); (b) beam paths.

principle used was to measure the Doppler shift in frequency of a frequency-modulated laser beam reflected from the lower face of the plate under investigation. This frequency shift is proportional to the velocity of the surface at the time of reflection, and a time integral of this velocity gives the displacement of the bottom surface. The laser beam used was modulated at a frequency of 2 MHz. This beam was produced by splitting a laser beam with a half mirror, modulating one of the resulting beams at 40 MHz and the other at 38 MHz and then recombining them, cf. Fig. 1(a).

This 2 MHz beam was now once again split into two plane polarized beams one of which was reflected at a fixed mirror, and the other was reflected from a mirrored surface painted onto the bottom face of the plate, the two beams were once again recombined, cf. Fig. 1(b). If the bottom face of the plate is moving, a Doppler shift of frequency proportional to its velocity takes place, and when it is recombined with the beam reflected from the fixed mirror, a beam modulated at a frequency proportional to this shift is produced. The magnitude of this shift can be determined with a frequency discriminator, and the displacement can, as mentioned above, be obtained as an integral of the particle velocity. (An alternative method of determining the displacement as long as it is less than $\lambda/4$ (where λ is the wavelength of the laser beam) is to determine the phase shift produced by the Doppler reflection, this is directly proportional to the displacement, and this method provides a better signal-to-noise ratio than the one which uses the frequency discriminator.)

This interferometer enabled us to obtain reliable displacement–time curves of the bottom faces of plates when steel balls of various diameters hit their top faces. We wanted, however, to show a separation between the two pulses which are produced by the impact of a ball, and which travel through the plate with the velocity of "P" waves and of "S" waves, respectively. In order for such a separation to be observable at the point on the bottom face of the plate immediately underneath the point of impact, the pulses had to be of very short duration, so that balls of very small diameter had to be used, and the plate had to be fairly thick. Unfortunately, we found that in order for the pulses then to be large enough to be recorded, the velocities of impact had to be relatively high, and at such high velocities of impact the glass plates fractured locally. Therefore, plastic plates had to be used instead. The plastic chosen was polystyrene, now while this plastic is viscoelastic, its specific damping capacity is very low, so that the attenuation due to viscoelastic effects does not change the wave forms appreciably. Figure 2 shows the records of displacements produced by impacts of spheres 4, 3, and 2 mm in diameter on a polystyrene plate, 59 mm thick. It may be seen in Fig. 2(a), the impact of a 4 mm diameter sphere, the two pulses are completely merged, in Fig. 2(b), the impact of a 3 mm sphere, separation is just beginning, while in Fig. 2(c), the impact of a 2 mm sphere, complete separation has taken place.

THEORETICAL

Consider an elastic plate of thickness h which is at rest for $t < 0$. Let a force $QH_0(t)$ be applied to a point on the upper face, then elastic waves will be propagated away from this point, and the nature of these waves can be determined by the method of Integral Transforms (cf. e.g. Achenbach[13] or Miklowitz[14]). In order to satisfy the condition of zero stress on the lower face of the plate, a system of reflected waves must be postulated, and these reflected waves travel back to the top surface where they are once again reflected. Thus, as time progresses, a whole series of waves reflected from the upper and lower faces of the plate have to be considered.

In the study of the rebound process, only waves in the immediate vicinity of the axis of symmetry normal to the plate are relevant, and the inverse transformations which are required can be found by the method of Cagniard[15]. This has been done by one of the present authors[16].

It was shown, that there is not only a "P" wave and an "S" wave of non-vanishing amplitude, propagating along the axis of symmetry, but when these waves are reflected at the bottom face of the plate, each wave produces two reflected waves, one "P" and one "S".

The vertical displacement of the point where the axis of symmetry meets the bottom face of the plate is shown in Fig. 3 as a function of time for a plate of Poisson's ratio 0.25 (e.g. glass), when a step function load is applied at the corresponding point on the top face of the plate.

The time τ is here measured in units of the transit time of an "S" wave through the thickness of the plate. Thus, the "P" wave arrives when $\tau = 1/\sqrt{3}$, while the "S" wave arrives at $\tau = 1$. As may be seen in the figure, the arrival of the "S" wave is accompanied by a sharp decrease in the slope of the curve. (The next wave which would arrive would be the "P" wave that had traversed the thickness of the plate three times, this would arrive at $\tau = \sqrt{3}$ but this arrival is not shown in the figure.)

In Fig. 3 the vertical displacement w is plotted in units of $Q/2\pi\mu h$, where w is the actual vertical displacement, and μ is the shear modulus of the material of the plate. In these units the initial jump in the displacement is $(c_2/c_1)^2$, where c_1 and c_2 are the velocities of "P" and "S" waves, respectively, in the material of the plate. Now, such an instantaneous jump in displacement is physically unrealistic, but it is just a consequence of the physically unrealistic assumption of a jump in the value of Q, which it is assumed is applied to a point on the top surface of the plate.

The reflected wave is shown in the same figure and it may be seen that the "P" wave front is reflected without change of amplitude. However, the amplitude of the reflected displacement behind the wave front is greater than that of the incident wave, and this enhancement of the reflected amplitude continues through the arrival and reflection of the "S" wave at $\tau = 1$.

Knopoff[17] carried out calculations similar to those just described, in a study of the propagation of a seismic disturbance in an elastic plate layer. He was, however, in his work, primarily concerned with disturbances the duration of which were short compared with the transit time of an "S" wave across the layer, and under such conditions the enhancement effect is not very important, and Knopoff did not pay much attention to it. This limitation does not, however, apply to the problems we are concerned with here, and we therefore have to consider it further. A possible physical interpretation of the enhancement effect will be discussed later in this paper.

As further reflections of the waves take place at the upper and lower faces of the plate, more and more waves are propagated in the plate. Now, while following these reflected waves is, in principle, reasonably straightforward analytically, the expressions become increasingly involved, and so we will confine our attention to the motion of the point of application of the force for the period of two reflections of the wave at the bottom face and two at the top face of the plate.

Figure 4 shows the *additional* normal displacement produced by these reflections at the point of application of the force. In this figure the displacement is plotted in units of $Q/2\pi\mu h$ as before, but the time scale τ' is now measured in units of $2h/c_2$ (i.e. it is half that

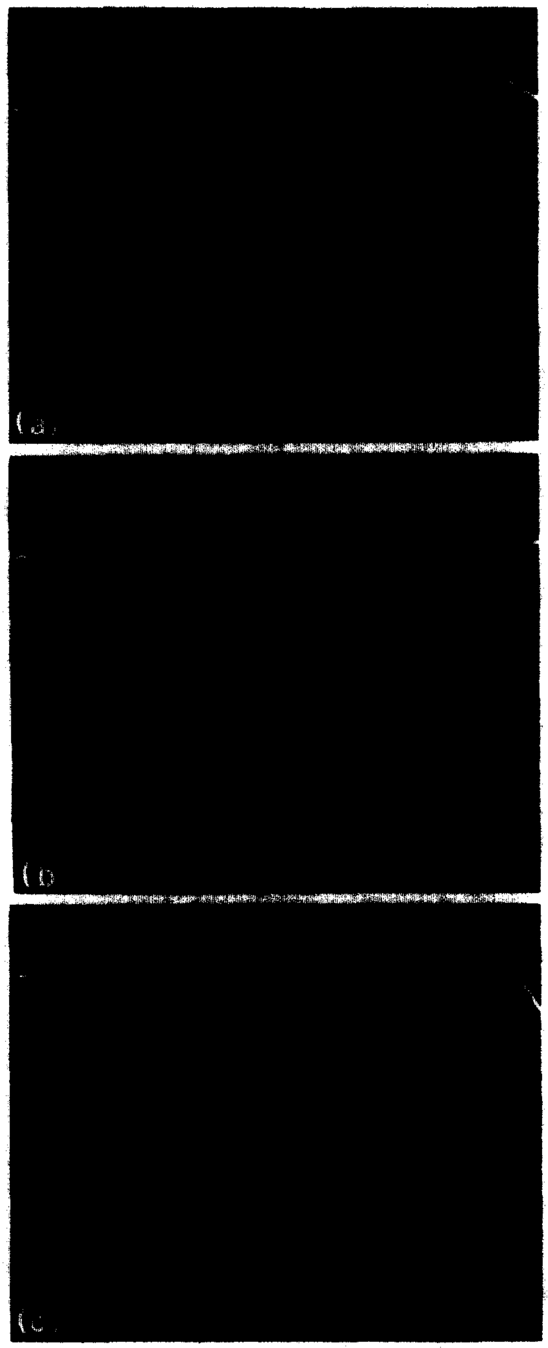

Fig. 2. Displacement–time records of the axial region of the bottom face of a polystyrene block 59 mm thick when the top face is hit by steel balls of different diameters: (a) 4 mm; (b) 3 mm; (c) 2 mm diameter ball.

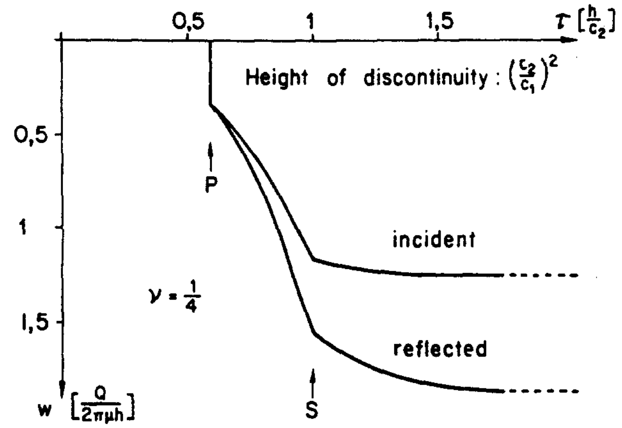

Fig. 3. Vertical displacement (Heaviside response) on the axis of symmetry at the bottom face. Incident and reflected wave.

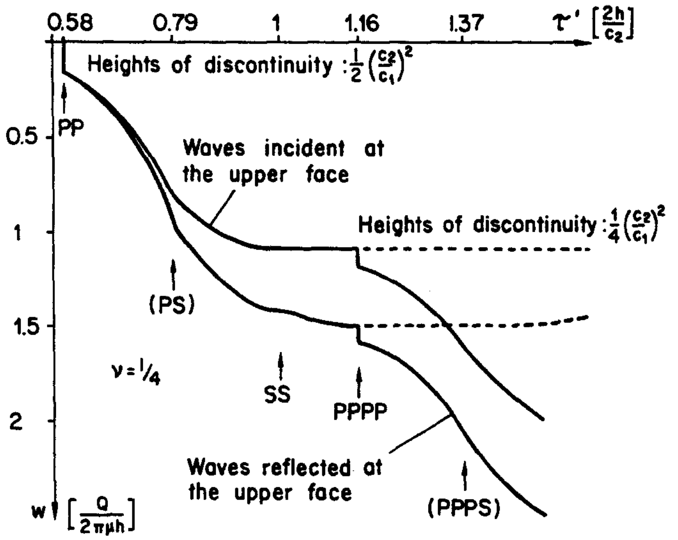

Fig. 4. The *additional* displacement (Heaviside response) at the point of load application due to the waves once and twice reflected at the bottom face.

used in Fig. 3). It may be seen that the first reflected wave to arrive is the "P" wave which had been reflected as a "P" wave at the bottom face of the plate. This wave arrives with a step front of amplitude $\frac{1}{2}(c_2^2/c_1^2)$ (since it has now travelled a distance $2h$) at $\tau' = 0.58$. Thus the wave front propagates as a spherically diverging acoustic wave, and its amplitude decays linearly with the distance of travel. Behind this front, however, the amplitude decays much more slowly as a result of the *enhancement effects* mentioned above.

Now although we have considered here only a very few reflections, it may be seen from the figure that the motion of the top face is already beginning to approximate to a linear increase of displacement with time as occurs in simple flexural motion, and thus it would seem not unreasonable to apply plate theory even when the duration of the loading is short enough for only a few reflections to take place during the impact process. In order to utilize the results of the analysis based on the response of a plate to a Heaviside function, to the problem with which we are concerned, namely the impact of a sphere on a plate, we need to evaluate the convolution integrals of the product of the step function response, and the derivative of the Hertzian impact force with respect to time.

COMPARISON OF THEORY AND EXPERIMENT

Experimental observations of the normal displacement of the axial region of the bottom face of a glass plate which had been subjected to an axial impact on its top face, were

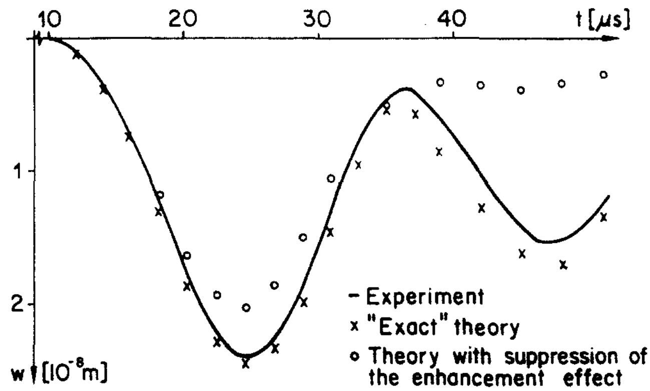

Fig. 5. Impact of a steel ball 5 mm in diameter on a glass plate 57 mm thick. Displacement–time history at the bottom face of the plate on the axis of symmetry. The two first arrivals of the wave due to the impact are shown.

compared with theoretically calculated values. These values were obtained by evaluating convolution integrals of the Heaviside response, as discussed in the previous section, and the experimental measurements were made using Doppler interference techniques which were described earlier.

The results of one such comparison are shown in Fig. 5. The impact considered is that of a steel ball 5 mm in diameter which has hit the top face of a glass plate 57 mm thick, the time interval covered in this figure goes from the arrival of the first "P" wave to the time when it has traversed the thickness of the plate three times and has once again arrived at the bottom face. It may be seen that the agreement between the experimental observation and the prediction of the "exact" theory is quite close.

For comparison, the predictions of a simplified theory are shown on the same figure. This treatment is based on the assumption that the "P" and "S" waves reflected from the bottom face may be regarded as spherically divergent pulses originating at the image of the point of impact on the bottom face of the plate. The reflected "P" and "S" pulses are assumed to be of opposite sign to the incident ones, but no *enhancement effects* have been taken into account. A somewhat similar treatment was used by Taylor in his analysis of the phenomenon of spalling produced by the detonation of small explosive charges on elastic plates[18]. It may be seen from Fig. 5 that, whereas this simplified treatment gives reasonable agreement with observations for times up to about the time of arrival of the "S" wave, large deviations are observed for later times, and the second maximum corresponding to the second arrival of the "P" pulse is almost totally absent. This figure highlights the importance of the *enhancement effect*, mentioned earlier, in describing the true motion of the plate.

A comparison was also made between theoretical predictions and experimental observations of the displacement for the impact of a ball 2 mm in diameter on a polystyrene plate 59 mm thick, the trace of which is shown in Fig. 2(c). This comparison is shown in Fig. 6 and it may be seen that the agreement is reasonably good, although the calculated values are somewhat larger than those observed experimentally. These differences are almost certainly attributable to the viscoelastic nature of the polystyrene. Although the internal friction in polystyrene is small, it is still appreciable, and both the "P" wave and the "S" wave will be subjected to viscoelastic losses. As shown by Lifshitz and Kolsky[19], the volume viscosity of these polymers is about one fifth of that for shear and this may account for the deviations being greater at the time of arrival of the "longitudinal" shear wave.

When we come to consider the loss of kinetic energy of the impinging sphere, ΔW, we see that this can be attributed to two effects. First there is the *Hunter loss* ΔW_1, which results from the energy carried away by the outgoing spherical compressive pulse, and secondly there is ΔW_2 the loss due to the setting up of flexural oscillations in the plate. Figure 7 shows a comparison between experimentally determined values of $1-e^2$, and the predicted values of $\Delta W_1 + \Delta W_2$ for the impact of steel spheres on glass plates. We can see from the figure that the experimentally determined loss is about 0.5% greater than that

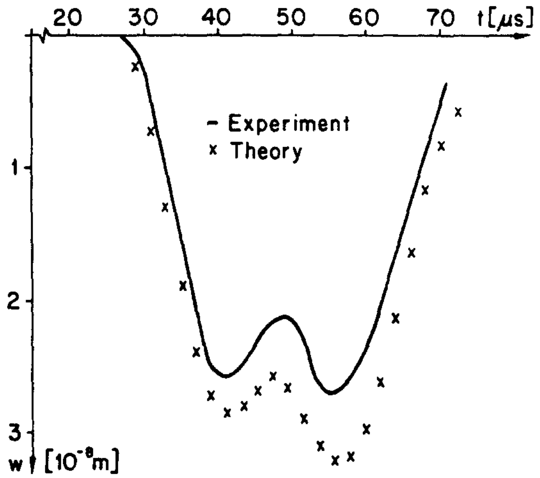

Fig. 6. Comparison of the experimental wave shown in Fig. 2(c) with theoretically obtained values (material damping ignored).

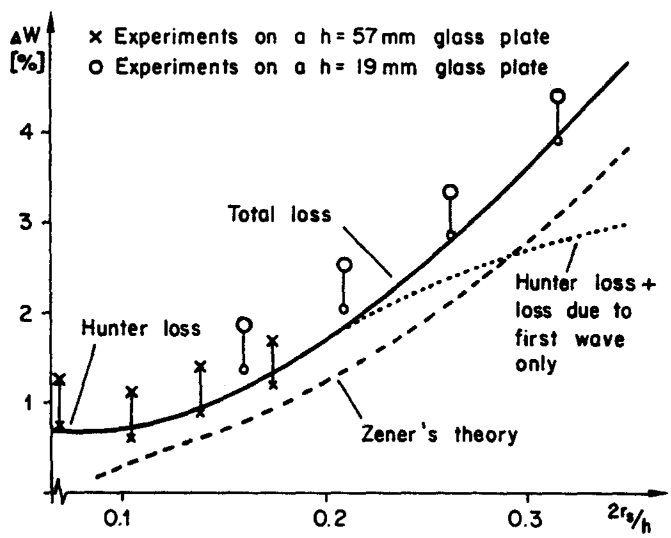

Fig. 7. Energy loss of the sphere during the impact on a very thick plate.

predicted theoretically. The reason for this deviation is not known, but friction between the impinging sphere and the plate, internal friction effects in the sphere and plate, and the kinetic energy used in setting up oscillations in the sphere all seem likely causes.

DISCUSSION

The foregoing mathematical analysis of the problem of the elastic waves generated by the impact of spheres on plates has led to the prediction of two effects which appear to be physically counter-intuitive. These effects are: (a) a longitudinal discontinuity which travels at the velocity of transverse elastic waves, and (b) an enhancement in amplitude of the incident longitudinal wave on its reflection at the free surface of the plate. It is proposed now to consider the physical bases of these two results further.

(a) *The apparently longitudinal* "S" *wave*

In the classical treatment of the propagation of elastic waves in unbounded isotropic media (e.g. Ref. [13]), it is shown that a discontinuity in particle velocity can travel at only one of two velocities. If the discontinuity is longitudinal, i.e. in the direction of propagation, it travels at the velocity of "P" waves, if on the other hand the discontinuity is transverse, i.e. perpendicular to the direction of propagation, it travels at the velocity of "S" waves.

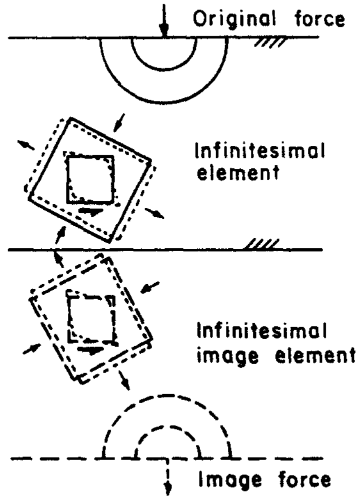

Fig. 8. Determination of the directions of the resultant shear stresses, due to the incident and the image waves obtained by a qualitative consideration of the *deviatoric* parts of the deformation.

Now, a careful examination of the derivation of this conclusion shows that what is considered is a discontinuity in $\dot{u}$, the longitudinal particle velocity or in $\dot{v}$, the tranverse particle velocity. Nothing is said in this proof about the effect of these discontinuities on the accelerations $\ddot{u}$ and $\ddot{v}$, for example no consideration is given to the possibility that a discontinuity in v could produce a discontinuity of lower order in $\dot{u}$, and we are suggesting that this in fact is what is happening here. Gakenheimer[20] has considered in detail the problem of a sudden load applied normally to a point on the free surface of an isotropic, elastic half space. He has shown that there is a discontinuity in the transverse component of the *displacement* at the front of the "S" wave and that this discontinuity decreases in magnitude with the size of the angle that the direction makes with the axis of symmetry. Along the symmetry axis the discontinuity disappears entirely. The longitudinal displacement, however, remains continuous across the "S" wave front. The longitudinal component of the *particle velocity*, however, shows a discontinuity at the "S" wave front, and this discontinuity persists even along the symmetry axis where the transverse component has vanished. It is perhaps worth noting that this secondary front decays much more rapidly than the front which is travelling at the "P" wave velocity, and in the far field it can no longer be observed.

(b) *The enhancement effect*

It has been shown analytically and verified experimentally that when the spherically divergent, longitudinal, stress pulse produced by an elastic impact is incident on a free surface, the longitudinal pulse reflected from the surface is everywhere except at the head of the pulse of higher amplitude than the incident pulse. We are proposing here to discuss the physical basis for this apparently anomalous behaviour.

In order to satisfy the stress free condition on the free surface we must postulate a reflected wave which produces stresses that exactly cancel out the stresses produced on that surface by the incident wave. Now in the acoustic case, where shear stresses cannot be sustained, this condition is met by a spherical wave of opposite sign which appears to diverge from the image point of the centre of the incident spherical wave in the free surface. In the case of a wave in an elastic solid however, the incident wave will produce shear stresses as well as normal stresses on the surface and the reflected wave of opposite sign will double the value of these shear stresses (cf. Fig. 8). Thus, in order to retain the stress free condition on the free surface, we have to assume that both a longitudinal reflected wave of opposite sign is reflected and at the same time a compensatory wave which will cancel out the shear stresses produced by the incident wave travels back into the solid.

To illustrate how this shear *compensation wave* will influence the amplitude of the reflected longitudinal wave, we postulate a drastically simplified model of the effect. We

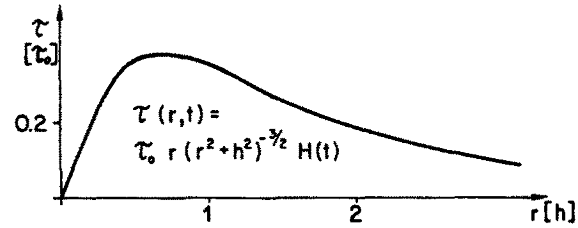

Fig. 9. Simplified shear distribution at the bottom face.

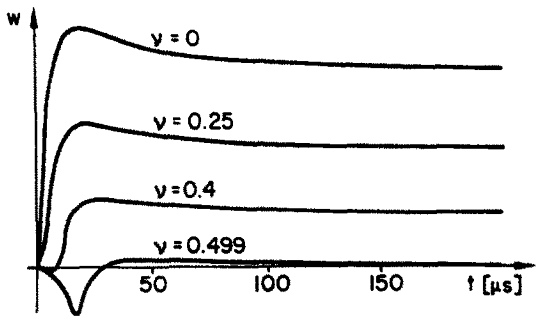

Fig. 10(a). Heaviside response: the ν-dependence of the difference of the displacement between the true reflected wave and the incident wave on the symmetry axis at the bottom face ($\mu/(3-2\nu)$ is kept constant).

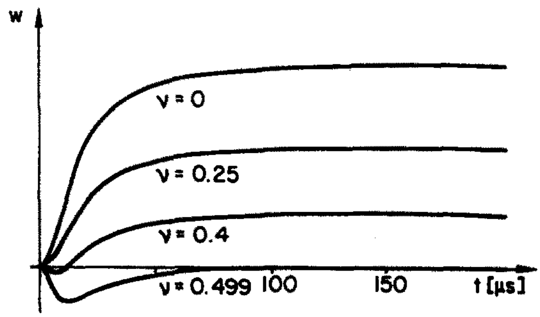

Fig. 10(b). The ν-dependence of the vertical displacement produced by the simplified "shear compensation wave" on the symmetry axis at the bottom face of the plate ($\mu/(3-2\nu)$ constant).

assume that the shear distribution at the bottom face to be compensated will be of the form shown in Fig. 9. This model ignores the differences in travel distances of the incident wave as it impinges on different points of the plate surface as well as the different times of arrival. Thus, consideration of the response of such a model might be expected to at best give only a qualitative indication of the observed differences in amplitude between the longitudinal displacements in the incident and reflected waves. Nevertheless, calculations were carried out for the amplitudes of the additional longitudinal displacement produced by this "τ"-compensation wave for different values of Poisson's ratio, ν, and these results were compared with the differences found for the exact theory. Thus, Fig. 10(a) shows the differences for the actual impact conditions as considered earlier in this paper while Fig. 10(b) shows the corresponding values produced by the postulated compensation wave. It may be seen in a general quantitative manner that the curves in the two cases are similar and that as $\nu \to 0.5$ the difference tends to disappear, since the behaviour is here approaching that of a fluid. The agreement is perhaps a little surprising in view of the drastic simplifications which were made to carry out this calculation, and perhaps it should be noted that the absence of the enhancement effect at the head of the pulse is a direct consequence of the fact that at this instant the amplitude of the "τ"-compensation wave is still zero.

CONCLUSION

This paper has discussed an experimental and theoretical treatment of the elastic impact of a sphere on an elastic plate which is neither so thick that it can be treated as a semi-infinite solid nor so thin that the impact can be regarded as producing simple flexural waves in the plate. The greater part of the analytical work has been concerned with the elastic waves set up in the plate, their propagation and reflection at the free surfaces of the plate and how a detailed study of this can lead to better understanding of the energy lost in setting up oscillations in the plate. In this study two counter-intuitive results have emerged, namely, along the axis of impact a discontinuity which travels at the velocity of transverse waves, and an enhancement of the amplitude of this pulse on its reflection at the bottom face of the plate. Both of these have been confirmed experimentally, and the physical bases of these anomalies have been discussed.

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